Ground state property of Bose-Einstein gas for arbitrary power low interaction

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Abstract

We study Bose-Einstein gas for an arbitrary power low interaction $C_\alpha r^{-\alpha}$. This is done by the Hartree Fock Bogoliubov (HFB) approach at $T \leq T_c$ and the mean field approach at $T > T_c$. Especially, we investigate the ground state property of Bose gas interacting through the Van der Waals $-C_6 r^{-6}$ plus $C_3 r^{-3}$ interactions. We show that the ground state under this interaction is stable if the ratio of coupling constants is larger than that of the critical curve. We find that the $C_3 r^{-3}$ term plays an important role for the stability of the ground state when the density of atoms becomes sufficiently large at low temperature. Further, using the numerical values of C_3 and C_6, we confirm that the ground state of alkali atoms are stable.

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1 Introduction

Bose-Einstein condensation (BEC) was first observed in dilute ultracold alkali atoms of rubidium [1], lithium [2] and sodium [3]. Furthermore, Mewes et al. and Ensher et al. measured the condensate fraction and the energy of rubidium atoms [4, 5]. It is shown that the transition temperature T_0 is shifted not only by the interaction effect but also by the finite size effect by a few percent, where T_0 denotes the transition temperature of the noninteracting Bose gas within the external field in the thermodynamic limit. This means that both the interaction and the finite size effects play an important role in the real Bose gases. In fact, the number of trapped atoms is typically $N \leq 10^7$, and this may not be sufficiently large to take the thermodynamic limit.

On the other hand, these experimental findings have stimulated much interest in the theory of the interacting Bose gas. Since BEC occurs when atoms are dilute and cold, we can treat the interaction of atoms as the two-body interaction. In this case, the s-wave scattering length characterizes the strength of the two-body interaction. Under these conditions, the Gross Pitaevskii (GP) equation can well describe the behavior of the interacting Bose gas at zero temperature [6, 7]. This is a mean field approach for the order parameter associated with the condensate. Using the GP equation, several authors studied the ground state and the excitation properties of the condensate [8]. To study BEC at finite temperature, the GP equation at zero temperature was extended by Griffin [9]. It is called the Hartree Fock Bogoliubov (HFB) theory. In particular, Popov approximation to the HFB theory has been employed to explain the experimental results [10].

In the realistic point of view, BEC experiments are carried out in magnetic traps. In this situation, spin-polarized atoms interact through a triplet po-

tential. As long as atoms remain polarized, they cannot form molecules. For alkali atoms, the triplet potential has many bound states which allow them to recombine into molecules. Since this recombination can only occur in a three-body scattering, it cannot occur for sufficiently low density of atoms. Thus, the two-body scattering is dominant. Therefore, spin-polarized atoms can remain the gas through dipole two-body scattering which flip the spin to untrapped state, and then atoms can produce BEC. In this sense, it is of particular importance to find a way to investigate BEC with more realistic interactions.

In this paper, we study BEC for an arbitrary power low interaction $C_\alpha r^{-\alpha}$ which is more realistic than the ordinary contact interaction (δ-function). This interaction plays an important role for alkali atoms [11]. The atomic interaction V of alkali atoms can be approximately written as [12]

$$V = V_c + V_d + V_{hf} + V_Z + V_{so}, \tag{1}$$

where V_c is the central force of the interaction. Further, V_c can be written in terms of the electron exchange interaction V_{ex} and the dispersion force $-C_6 r^{-6} - C_8 r^{-8} - C_{10} r^{-10} \cdots$. The magnetic-dipole interaction V_d and the hyperfine interaction V_{hf} behave like $C_3 r^{-3}$, where the coupling constant C_3 is the spin part of the interaction. The last two terms, V_Z and V_{so} represent the Zeeman interaction and the spin-orbit interaction, respectively. These interactions are important at high density.

Here, we study Bose gas interacting through the Van der Waals $-C_6 r^{-6}$ plus $C_3 r^{-3}$ interactions. We show that the ground state under this interaction is stable if the ratio of the coupling constants is larger than that of the critical curve. We find that the $C_3 r^{-3}$ term plays an important role to stabilize the ground state when the density of atoms becomes large at low temperature. Using the numerical values of C_3 and C_6, we confirm that the ground state

of alkali atoms are stable.

This paper is organized in the following way. In the next section, we derive the effective Hamiltonian for the interacting Bose gas. Then, in section 3, we study the ground state stability for the Van der Waals plus $C_3 r^{-3}$ interactions. Section 4 summarizes what we have clarified in this paper.

2 The Effective Hamiltonian for Interacting Bose Gas

In this section, we derive the effective Hamiltonian for the interacting Bose gas. Since Bose gas dramatically changes its behavior below the transition temperature T_c, we derive the effective Hamiltonian with the HFB approach at $T \leq T_c$. On the other hand, we derive the effective Hamiltonian with the mean field approach at $T > T_c$.

The Hamiltonian for the interacting Bose gas confined in a harmonic oscillator potential can be written as

$$\begin{aligned} \hat{H} &= \int d\mathbf{r}\hat{\Psi}^\dagger(\mathbf{r})\left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega r^2\right]\hat{\Psi}(\mathbf{r}) \\ &\quad + \frac{1}{2}\int d\mathbf{r}_1 d\mathbf{r}_2 \hat{\Psi}^\dagger(\mathbf{r}_1)\hat{\Psi}^\dagger(\mathbf{r}_2)V(|\mathbf{r}_1 - \mathbf{r}_2|)\hat{\Psi}(\mathbf{r}_2)\hat{\Psi}(\mathbf{r}_1), \end{aligned} \tag{2}$$

where $\hat{\Psi}(\mathbf{r})$ is the boson field operator. The two-body atomic interaction $V(|\mathbf{r}_1 - \mathbf{r}_2|)$ is given as

$$V(|\mathbf{r}_1 - \mathbf{r}_2|) = \frac{C_\alpha}{|\mathbf{r}_1 - \mathbf{r}_2|^\alpha}. \tag{3}$$

Now, we write down the second quantized Hamiltonian for Eq. (2) [13]. The boson field operator for the ideal system can be expanded by plane waves, but for the general case, the corresponding field operator is a sum over all normal modes [14]

$$\hat{\Psi}(\mathbf{r}) = \sum_\nu \hat{a}_\nu \chi_\nu(\mathbf{r}), \tag{4}$$

when $\chi_\nu(\mathbf{r})$'s are any complete set of normalized single-particle wave function, and a_ν is a bosonic annihilation operator for the single-particle state ν. From Eq. (4), the Hamiltonian can be written as

$$\hat{H} = \sum_{\nu\nu'} T_{\nu\nu'} \hat{a}^\dagger_\nu \hat{a}_{\nu'} + \frac{1}{2} \sum_{\nu\lambda\nu'\lambda'} V^{\nu\lambda}_{\nu'\lambda'} \hat{a}^\dagger_\nu \hat{a}^\dagger_\lambda \hat{a}_{\nu'} \hat{a}_{\lambda'}, \tag{5}$$

with

$$T_{\nu\nu'} = \int d\mathbf{r} \chi^\dagger_\nu(\mathbf{r}) \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\rm ext}(\mathbf{r}) \right] \chi_{\nu'}(\mathbf{r}), \tag{6}$$

$$V^{\nu\lambda}_{\nu'\lambda'} = \int d\mathbf{r}_1 d\mathbf{r}_2 \chi^\dagger_\nu(\mathbf{r}_1) \chi^\dagger_\lambda(\mathbf{r}_2) V(|\mathbf{r}_1 - \mathbf{r}_2|) \chi_{\nu'}(\mathbf{r}_2) \chi_{\lambda'}(\mathbf{r}_1). \tag{7}$$

Here, we expand $V(|\mathbf{r}_1 - \mathbf{r}_2|)$ in terms of the Legendre polynomial P_l

$$\begin{aligned} V(|\mathbf{r}_1 - \mathbf{r}_2|) &= \sum_{l=0}^{\infty} v_l(r_1, r_2) P_l(\cos\theta_{12}), \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} v_l(r_1, r_2) Y^*_{l,m}(\theta_1, \varphi_1) Y_{l,m}(\theta_2, \varphi_2), \end{aligned} \tag{8}$$

where $v_l(r_1, r_2)$ is given by

$$v_l(r_1, r_2) = \frac{2l+1}{2} \int_{-1}^{1} dt \; V(|\mathbf{r}_1 - \mathbf{r}_2|) P_l(t). \tag{9}$$

In the case of the power low potential $V(|\mathbf{r}_1 - \mathbf{r}_2|) = C_\alpha |\mathbf{r}_1 - \mathbf{r}_2|^{-\alpha}$, we can integrate $v_l(r_1, r_2)$;

$$v_l(r_1, r_2) = C_\alpha \frac{\Gamma(\frac{1}{2})\Gamma(l + \frac{\alpha}{2})}{\Gamma(l + \frac{1}{2})\Gamma(\frac{\alpha}{2})} \frac{r_2^l}{r_1^{\alpha+l}} \; {}_2F_1\left(\frac{\alpha}{2} + l, \frac{\alpha}{2} - \frac{1}{2}, l + \frac{3}{2}, \frac{r_2^2}{r_1^2} \right), \tag{10}$$

for $r_1 > r_2$, and

$$v_l(r_1, r_2) = C_\alpha \frac{\Gamma(\frac{1}{2})\Gamma(l + \frac{\alpha}{2})}{\Gamma(l + \frac{1}{2})\Gamma(\frac{\alpha}{2})} \frac{r_1^l}{r_2^{\alpha+l}} \; {}_2F_1\left(\frac{\alpha}{2} + l, \frac{\alpha}{2} - \frac{1}{2}, l + \frac{3}{2}, \frac{r_1^2}{r_2^2} \right), \tag{11}$$

for $r_2 > r_1$, where ${}_2F_1(a, b, c, x)$ denotes hypergeometric function. Thus, we obtain the second quantized Hamiltonian for Bose gas interacting through the power low interaction. From now on, we present the mean field Hamiltonian and the HFB Hamiltonian.

2.1 Mean Field Theory

First, we present the mean field theory for the interacting Bose gas [15]. This theory is particularly valid for $T > T_c$ [16]. But for $T \leq T_c$, this theory is inadequate, since the low energy part of the excited states plays an important role at low temperature. This effective Hamiltonian $\hat{H}_{eff}$ is given by the diagonal part of the Hamiltonian (5), and can be written as

$$\hat{H}_{eff} = \sum_{\nu} T_{\nu\nu}\hat{n}_{\nu} + \frac{1}{2}\sum_{\nu\nu'} v_{\nu\nu'}\hat{n}_{\nu}\hat{n}_{\nu'}, \tag{12}$$

where $n_{\nu} = a_{\nu}^{\dagger}a_{\nu}$ and

$$v_{\nu\nu} = \int d\mathbf{r}_1 d\mathbf{r}_2 |\chi_{\nu}(\mathbf{r}_1)|^2 V(|\mathbf{r}_1 - \mathbf{r}_2|)|\chi_{\nu}(\mathbf{r}_2)|^2, \tag{13}$$

for $\nu = \nu'$ and

$$\begin{aligned} v_{\nu\nu'} &= \int d\mathbf{r}_1 d\mathbf{r}_2 |\chi_{\nu}(\mathbf{r}_1)|^2 V(|\mathbf{r}_1 - \mathbf{r}_2|)|\chi_{\nu'}(\mathbf{r}_2)|^2 \\ &\quad + \int d\mathbf{r}_1 d\mathbf{r}_2 \chi_{\nu}^{\dagger}(\mathbf{r}_1)\chi_{\nu'}^{\dagger}(\mathbf{r}_2) V(|\mathbf{r}_1 - \mathbf{r}_2|)\chi_{\nu}(\mathbf{r}_2)\chi_{\nu'}(\mathbf{r}_1) \end{aligned} \tag{14}$$

for $\nu \neq \nu'$. We rewrite Eq. (12) in the following way

$$\hat{H}_{eff} = \hat{H}_0 + \hat{H}', \tag{15}$$

with

$$\hat{H}_0 = \sum_{\nu} \tilde{E}_{\nu}\hat{n}_{\nu} - \frac{1}{2}\sum_{\nu\nu'} v_{\nu\nu'}\rho_{\nu}\rho_{\nu'}, \tag{16}$$

$$\hat{H}' = \frac{1}{2}\sum_{\nu\nu'} v_{\nu\nu'}(\hat{n}_{\nu} - \rho_{\nu})(\hat{n}_{\nu'} - \rho_{\nu'}), \tag{17}$$

$$\tilde{E}_{\nu} = T_{\nu\nu} + \sum_{\nu'} v_{\nu\nu'}\rho_{\nu'}. \tag{18}$$

Here, we minimize $\hat{H}'$ by taking ρ's in the following way

$$\rho_{\nu} = \langle n_{\nu} \rangle = \frac{\mathrm{Tr}\ n_{\nu}\ e^{-(\hat{H}_0 - \mu\hat{N})/k_B T}}{\mathrm{Tr}\ e^{-(\hat{H}_0 - \mu\hat{N})/k_B T}} = \frac{1}{e^{(\tilde{E}_{\nu} - \mu)/k_B T} - 1}. \tag{19}$$

Then, the thermodynamic potential $\Omega = -pV$ can be given as

$$\begin{aligned}\Omega &= -k_B T \ln \mathrm{Tr}\ e^{-(\hat{H}_0 - \mu\hat{N})/k_B T}, \\ &\simeq -\frac{1}{2}\sum_{\nu\nu'} v_{\nu\nu'}\langle n_\nu\rangle\langle n_{\nu'}\rangle + k_B T \sum_{\nu'} \ln\left(1 - e^{-(\tilde{E}_{\nu'} - \mu)/k_B T}\right). \end{aligned} \tag{20}$$

We note that Eq. (19) is also determined by the condition

$$\left(\frac{\partial\Omega}{\partial\langle n_\nu\rangle}\right)_{T,\mu,V,\langle n_{\nu'}\rangle\neq\langle n_\nu\rangle} = -\sum_{\nu'} v_{\nu\nu'}\langle n_{\nu'}\rangle + \sum_{\nu'} v_{\nu\nu'}\frac{1}{e^{(\tilde{E}_{\nu'}-\mu)/k_B T} - 1} = 0. \tag{21}$$

This condition means that the thermodynamic potential Ω is an extremum with respect to any change of T, V and μ in a state of thermal equilibrium.

2.2 HFB Theory

Now, we present the HFB theory. This theory is reliable for the description of the low temperature behavior of the Bose gas. In this theory, the operators a_0 and $a_0^\dagger$ are replaced by the c-number $a_0, a_0^\dagger \approx \sqrt{N_0}$. Then, the HFB Hamiltonian $\hat{K} = \hat{H} - \mu\hat{N}$ is given by

$$\begin{aligned}\hat{K} &= (T_{00} - \mu)N_0 + \frac{1}{2}V_{00}^{00}N_0^2 \\ &\quad + \sum_{\nu\nu'\neq 0}\left\{T_{\nu\nu'} - \mu\delta_{\nu\nu'} + 2N_0 V_{0\nu'}^{0\nu} + 2n'_\lambda V_{\lambda\nu'}^{\lambda\nu}\right\} a_\nu^\dagger a_{\nu'} \\ &\quad + \frac{N_0}{2}\sum_{\nu\nu'\neq 0} V_{\nu\nu'}^{00}(a_\nu a_{\nu'} + a_\nu^\dagger a_{\nu'}^\dagger), \end{aligned} \tag{22}$$

where n'_λ is the expectation value of the non-condensate density of particles, and we eliminate the lower power of N_0 and terms which are proportional to a_ν and $a_\nu^\dagger$. Now, we define new boson operators by Bogoliubov transformation

$$c_\nu = u_\nu a_\nu + v_\nu a_\nu^\dagger, \tag{23}$$

$$c_\nu^\dagger = u_\nu a_\nu^\dagger + v_\nu a_\nu, \tag{24}$$

where u_ν and v_ν satisfy $u_\nu^2 - v_\nu^2 = 1$. Then, we can write the Hamiltonian in terms of the new operator c_ν

$$\begin{aligned}\hat{K} &= (T_{00}-\mu)N_0 + \frac{1}{2}V_{00}^{00}N_0^2 + \sum_{\nu\nu'\neq 0}\left(\varepsilon_{\nu\nu'}v_\nu^2 - N_0 V_{\nu\nu'}^{00}u_\nu v_{\nu'}\right) \\ &+ \sum_{\nu\nu'\neq 0}\Bigg[\left\{\varepsilon_{\nu\nu'}(u_\nu u_{\nu'} + v_\nu v_{\nu'}) - N_0 V_{\nu\nu'}^{00}(u_\nu v_{\nu'} + v_\nu u_{\nu'})\right\}c_\nu^\dagger c_{\nu'} \\ &+ \left\{-\varepsilon_{\nu\nu'}u_\nu v_{\nu'} + \frac{1}{2}N_0 V_{\nu\nu'}^{00}(u_\nu u_{\nu'} + v_\nu v_{\nu'})\right\}(c_\nu^\dagger c_{\nu'}^\dagger + c_\nu c_{\nu'})\Bigg], \quad (25)\end{aligned}$$

where $\varepsilon_{\nu\nu'}$ is given as

$$\varepsilon_{\nu\nu'} = T_{\nu\nu'} - \mu\delta_{\nu\nu'} + 2\left(N_0 V_{\nu\nu'}^{00} + n'_\lambda V_{\lambda\nu'}^{\lambda\nu}\right). \quad (26)$$

We can eliminate the terms proportional to $c_\nu^\dagger c_{\nu'}^\dagger + c_\nu c_{\nu'}$ by imposing the condition

$$-\varepsilon_{\nu\nu'}u_\nu v_{\nu'} + \frac{1}{2}N_0 V_{\nu\nu'}^{00}(u_\nu u_{\nu'} + v_\nu v_{\nu'}) = 0, \quad (27)$$

or

$$\varepsilon_{\nu\nu'}u_{\nu'} - N_0 V_{\nu\nu'}^{00}v_{\nu'} = E_{\nu'}u_{\nu'}\delta_{\nu\nu'}, \quad (28)$$

$$\varepsilon_{\nu\nu'}v_{\nu'} - N_0 V_{\nu\nu'}^{00}u_{\nu'} = -E_{\nu'}v_{\nu'}\delta_{\nu\nu'}, \quad (29)$$

where the eigenvalue E_ν is given as

$$E_\nu = \sqrt{\varepsilon_{\nu\nu}^2 - \left(N_0 V_{\nu\nu}^{00}\right)^2}. \quad (30)$$

Then, we obtain the diagonal Hamiltonian

$$\hat{K} = (T_{00}-\mu)N_0 + \frac{1}{2}V_{00}^{00}N_0^2 + \sum_{\nu\neq 0}E_\nu c_\nu^\dagger c_\nu. \quad (31)$$

Here, we note that the first two terms in the right hand side of Eq. (31) correspond to the GP equation, and the third term in the right hand side of Eq. (25) represents small correction terms, and therefore we can ignore the

third term. From Eq. (31), we can also obtain the thermodynamic potential

$$\begin{aligned}\Omega &= -k_B T \ln \mathrm{Tr} e^{-\hat{K}/k_B T} \\ &\simeq (T_{00} - \mu) N_0 + \frac{1}{2} V_{00}^{00} N_0^2 + k_B T \sum_{\nu \neq 0} \ln(1 - e^{-E_\nu / k_B T}). \end{aligned} \tag{32}$$

The chemical potential is given by the condition

$$\begin{aligned} 0 &= \frac{\partial \Omega}{\partial N_0}, \\ &= T_{00} - \mu + V_{00}^{00} N_0 + 2 \sum_{\nu \neq 0} n'_\nu V_{\nu\nu}^{00}, \end{aligned} \tag{33}$$

where we ignore the anomalous average.

We note that Eqs. (31) and (33) are similar to results of the HFB approach to the Popov approximation [9]. In the next section, we will see the ground state stability of the interacting Bose gas using the GP equation.

3 Ground State Stability

In this section, we study the ground state properties of the Bose gas interacting through the Van der Waals type $-C_6 r^{-6}$ and the $C_3 r^{-3}$ interactions. We note that we must introduce a cut off at $r = 2\langle R \rangle$, since these interactions diverge at $r = 0$. Here, we assume the hard core potential inside the atomic radius $\langle R \rangle$. This is chosen to be the exponential or $C_{12} r^{-12}$ potential [12].

Now, we study the ground state stability of Bose gas for the condensate state at $T \leq T_c$. In this case, the HFB theory is reliable for the investigation of the condensate state. To investigate the ground state stability, we employ a variational method. It is a good approximate scheme to study the ground state stability. We assume the following Gaussian wave function for the ground state

$$\Psi(\mathbf{r}) = \sqrt{\frac{N}{\pi^{3/2} \sigma^3}} e^{-r^2/2\sigma^2}, \tag{34}$$

where σ represents the variational parameter. This choice is natural when we take the noninteracting limit.

First, we consider the Van der Waals interaction. Here, we make comments on the scattering length a for this interaction. This can be analytically given by [17]

$$a = \frac{\Gamma(3/4)}{2\sqrt{2}\Gamma(5/4)}\left(\frac{mC_6}{\hbar^2}\right)^{\frac{1}{4}}\left[1-\tan\left(\phi-\frac{\pi}{8}\right)\right], \tag{35}$$

where ϕ is the semiclassical phase calculated at zero energy from the classical turning point to infinity. This phase depends on the repulsive hard core potential. Substituting Eq. (34) into Eq. (31), we can obtain the ground state energy E_{g_6}

$$\begin{aligned} E_{g_6} &= \frac{3}{4}N\hbar\omega(\sigma^2+\sigma^{-2}) - \frac{8N^2C_6}{\pi a_{ho}^6\sigma^6}I_6, \\ &= \frac{3}{2}N\hbar\omega\left\{\frac{1}{2}(\sigma^2+\sigma^{-2}) - g_6\sigma^{-6}\right\}, \end{aligned} \tag{36}$$

where we rewrite σ as $\sigma = \sigma/a_{ho}$ in units of the harmonic oscillator length $a_{ho} = \sqrt{\hbar/m\omega}$ and I_6 denotes the dimensionless integral

$$I_6 = 2\int_0^\infty ds\ s^2e^{-s^2}\int_{s+2\langle\tilde{R}\rangle}^\infty dt\ t^2e^{-t^2}\frac{1}{t^6}\frac{1+s^2/t^2}{(1-s^2/t^2)^4}. \tag{37}$$

The dimensionless coupling constant g_6 is defined by

$$g_6 = \frac{16NC_6I_6}{3\pi\hbar\omega a_{ho}^6}. \tag{38}$$

Table 1. We plot the integral values of I_3 and I_6 for several cases of $2\langle\tilde{R}\rangle$.

$2\langle\tilde{R}\rangle$	I_3	I_6
1.0×10^{-4}	1.295	1.306×10^{10}
2.0×10^{-4}	1.187	1.632×10^{9}
3.0×10^{-4}	1.123	4.835×10^{8}
4.0×10^{-4}	1.078	2.040×10^{8}
5.0×10^{-4}	1.043	1.044×10^{8}
6.0×10^{-4}	1.015	6.044×10^{7}
7.0×10^{-4}	0.991	3.806×10^{7}
8.0×10^{-4}	0.970	2.550×10^{7}
9.0×10^{-4}	0.951	1.791×10^{7}
1.0×10^{-3}	0.935	1.306×10^{7}

In Table 1, we plot the values of I_6 as a function of the dimensionless parameter $2\langle\tilde{R}\rangle = 2\langle R\rangle/a_{ho}$. We note that $2\langle R\rangle$ and a_{ho} are of the order of $\mathring{A}$ and μm, respectively. Therefore, $2\langle\tilde{R}\rangle$ is of the order of 10^{-4}.

Now, we look for the critical coupling constant $\tilde{g}_6$. This is given by the inflexion point of E_{g_6}

$$\frac{dE_{g_6}}{d\sigma} = \frac{d^2E_{g_6}}{d\sigma^2} = 0, \tag{39}$$

at $\sigma = \sigma_c$ and $g_6 = \tilde{g}_6$. From Eq. (36), we obtain

$$\tilde{g}_6 \;=\; \frac{1}{24} \sim 0.0417. \tag{40}$$

Next, we consider the C_3r^{-3} interaction [18]. We can write down the ground state energy E_{g_3}

$$\begin{aligned} E_{g_3} &= \frac{3}{4}N\hbar\omega(\sigma^2+\sigma^{-2}) + \frac{8N^2C_3}{\pi a_{ho}^3\sigma^3}I_3, \\ &= \frac{3}{2}N\hbar\omega\left\{\frac{1}{2}(\sigma^2+\sigma^{-2}) + g_3\sigma^{-3}\right\}, \end{aligned} \tag{41}$$

where I_3 is also a dimensionless integral defined as

$$I_3 \;=\; 2\int_0^\infty ds\; s^2e^{-s^2}\int_{s+2\langle\tilde{R}\rangle}^\infty dt\; t^2e^{-t^2}\frac{1}{t^3(1-s^2/t^2)} \tag{42}$$

In Table 1, we also plot the values of I_3 as a function of $2\langle\tilde{R}\rangle$. The dimensionless coupling constant g_3 is defined by

$$g_3 = \frac{16NC_3I_3}{3\pi\hbar\omega a_{ho}^3}. \tag{43}$$

In this case, the ground state energy has the same property as the contact interaction [19]. We can obtain the critical coupling constant $\tilde{g}_3 \sim -0.178$. Finally, we consider the $C_3r^{-3} - C_6r^{-6}$ interaction. Since the Van der Waals interaction is weak, we always assume $|g_3| > g_6 \geq 0$. The ground state energy E_g is given by

$$E_g = \frac{3}{2}N\hbar\omega\left\{\frac{1}{2}(\sigma^2+\sigma^{-2}) + g_3\sigma^{-3} - g_6\sigma^{-6}\right\}. \tag{44}$$

In Fig. 1, we show the ground state energy in units of $3/2N\hbar\omega$ as a function

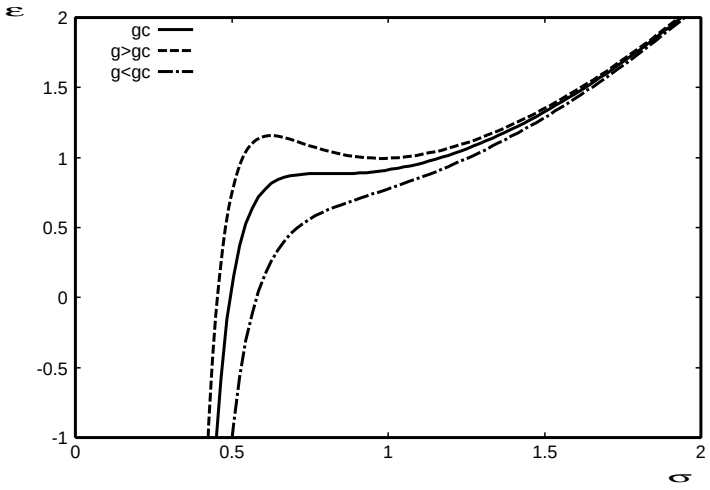

Figure 1: We show the ground state energy Eq. (44) in units of $3/2N\hbar\omega$ as a function of σ for several values of the parameter $g = g_3/g_6$. The solid line is drawn with $g = g_c$. The dashed and dot-dashed lines are $g > g_c$ and $g < g_c$, respectively.

of σ for several values of the parameter $g = g_3/g_6$ which is given as

$$g = \frac{C_3I_3}{C_6I_6}a_{ho}^3. \tag{45}$$

As can be seen, for $g > g_c$, the ground state energy always has a minimum. Therefore, the ground state is stable. On the other hand, for $g < g_c$, a minimum disappears. In this case, the ground state is unstable, therefore, BEC collapses. We can also calculate the critical value $g_c = \tilde{g}_3/\tilde{g}_6$. It is given by

$$9\tilde{g}_3 = 8\sigma_c^5 - 4\sigma_c, \tag{46}$$

$$18\tilde{g}_6 = 5\sigma_c^8 - \sigma_c^4. \tag{47}$$

Coupling constants $\tilde{g}_3$ and $\tilde{g}_6$ are related to each other through Eqs. (46) and (47). We note that the parameter σ_c must be $\sigma_c \geq 0.66874$, since we assume $g_6 \geq 0$. Eliminating σ_c from Eqs. (46) and (47), we obtain

$$\tilde{g}_3 = \frac{8}{9}\left(\frac{1+\sqrt{1+360\tilde{g}_6}}{10}\right)^{5/4} - \frac{4}{9}\left(\frac{1+\sqrt{1+360\tilde{g}_6}}{10}\right)^{1/4}. \tag{48}$$

In Fig. 2, we show the phase diagram of coupling constants g_3 and g_6. The

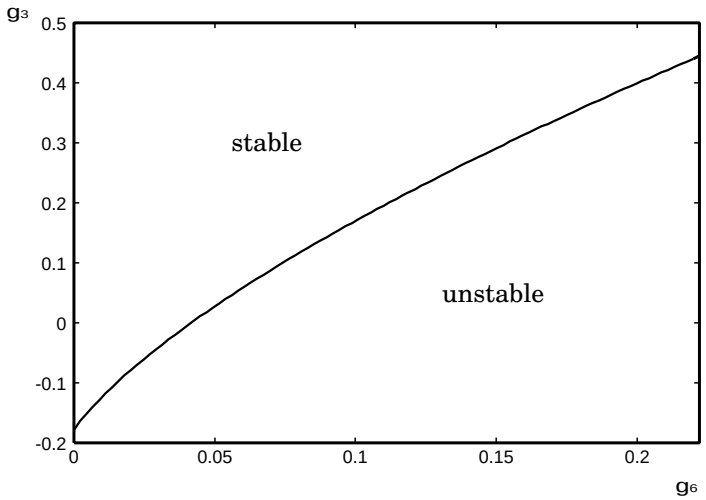

Figure 2: We show the phase diagram of coupling constants g_3 and g_6. The curve corresponds to the critical curve given by Eq. (48). The parameter runs $0.66874 \leq \sigma_c \leq 1$ in this diagram. Above the critical curve, the ground state energy is stable, and below the critical curve, it is unstable.

curve corresponds to the critical curve given by Eq. (48). Above the critical

curve, the ground state energy is stable, and below the critical curve, it is unstable. Therefore, it is seen that the BEC collapses below the critical curve in the phase diagram.

Now, we estimate the coupling constant g for alkali atoms. Coefficients C_3 and C_6 are numerically given by Ref. [20] for alkali atoms.

Table 2. We plot the ratio $|C_3|/C_6 \times 10^3$ between S and $P\sigma$ and$P\pi$ state for alkali atoms in units of the atomic units (a.u.).

	Li	Na	K	Rb	Cs
S-$P\sigma$	5.329	2.995	1.845	1.527	1.205
S-$P\pi$	3.298	2.325	1.377	1.144	0.885

In Table 2, we plot the ratio $|C_3|/C_6$ for alkali atoms in units of the atomic units (a.u.). As can be seen from Table 2, $|C_3|/C_6$ is of the order of 10^{-3} a.u.. The ratio I_3/I_6 and the harmonic oscillator length a_{ho} are of the order of $10^{-7} \sim 10^{-8}$ and 10^4 a.u., respectively. Thus, the coupling constant g is of the order of $10 \sim 10^2$. In Fig. 3, we show the critical coupling constant

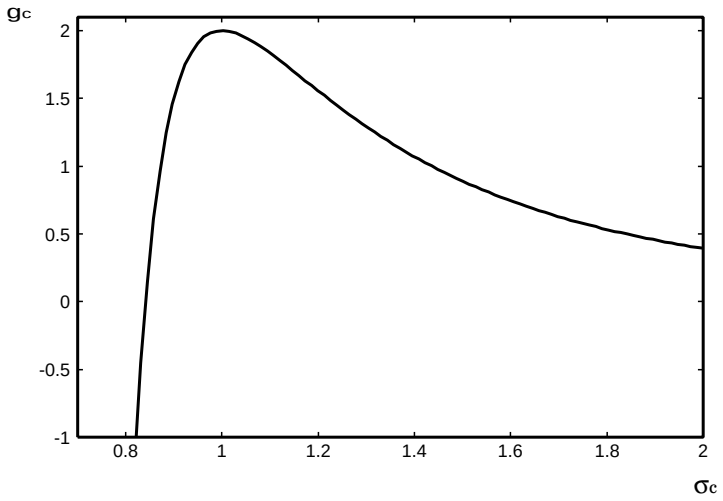

Figure 3: We show the critical coupling constant g_c as a function of σ_c. Above the curve, the ground state energy is stable, and below the curve, it is unstable.

g_c as a function of σ_c. Above the curve, the ground state is stable, and below the curve, it is unstable. As can be seen from Fig. 3, for $g > 0$, the

ground state of alkali atoms are always stable in our calculation. On the other hand, for $g < 0$, we must carefully consider the ground state stability, since it depends on σ_c. We can show that the ground state is stable when $\sigma_c \leq 0.756$ for $g = -10$, and when $\sigma_c \leq 0.690$ for $g = -10^2$. For real atoms, the value of σ_c is expected to be the same as the δ-function type interaction $\sigma_c = 0.669$ [19]. Therefore, the ground state is also stable for $g < 0$.

4 Conclusions

We have studied the ground state stability of interacting Bose-Einstein gas with an arbitrary power low interaction. In particular, we have considered the Van der Waals $-C_6 r^{-6}$ plus $C_3 r^{-3}$ interactions. Using the Gaussian variational function, we have obtained the ground state energy. It is shown that the ground state stability depends on the ratio of the coupling constants g_3 and g_6, then the critical coupling constant is obtained by the inflexion point of the ground state energy. For $g > g_c$, we can always produce stable BEC. On the other hand, for $g < g_c$, BEC collapses. Here, the ground state stability mainly depends on the $C_3 r^{-3}$ term, since atoms become sensitive to this interaction when the density of atoms becomes high at low temperature. Therefore, we can understand that the role of this interaction is important for the stability of the ground state at low temperature.

Next, we have obtained the phase diagram of coupling constants g_3 and g_6. From the phase diagram, we have shown that there exists a stable region for Bose-Einstein gas. It is useful because we can classify atoms which become BEC.

Finally, we have argued the validity of these results by using the numerical values of C_3 and C_6 for alkali atoms. We have shown that the ground state

of alkali atoms are always stable.

Our results show that we can reliably understand Bose-Einstein gas interacting through the realistic interaction beyond the GP theory.

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